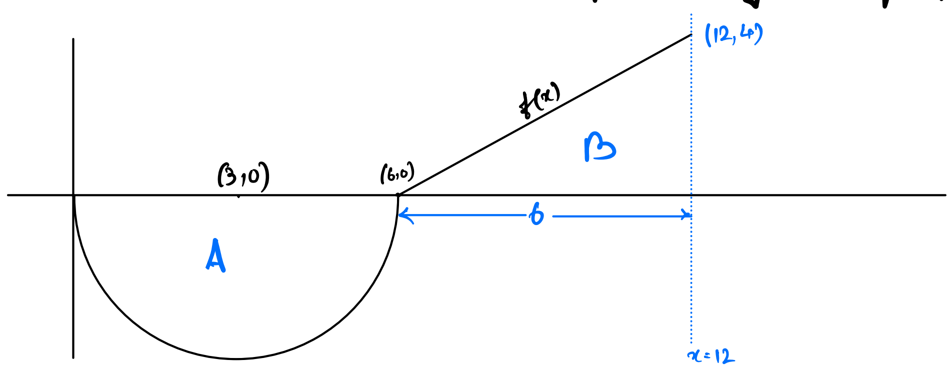
Review: Midterm #3

- Applied Optimization
- Increasing / Decreasing, Concave Up / Down.
- Antiderivatives
- Riemann Sums
- Integration

Calculate: $\int_0^{12} f(x)\,dx$, where $f(x)$ is given by the graph



Here, area A = (area of the semi-circle with radius 3)

$$= \left(\frac{\pi(3)^2}{2}\right)$$

$$= \frac{9\pi}{2}$$

area B = area of the right triangle, whose base is 6 & height 4

$$= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(6)(4) = 12$$

By sign rule of area we get,

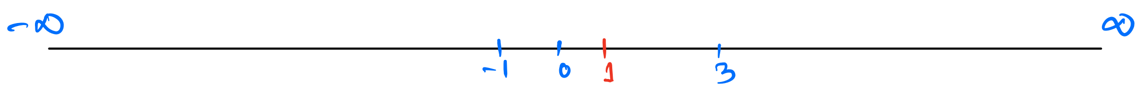
$$\int_0^{12} f(x)\,dx = -A + B = \left(-\frac{9\pi}{2} + 12\right)$$

Q. For $f(x) = x^3 - 3x^2 - 9x + 4$, find the intervals where f is increasing/decreasing, concave up/down & find the points of local maxima/minima/point of inflections. Also find the maximum & minimum values of f.

$\rightsquigarrow$ Here, $f(x) = x^3 - 3x^2 - 9x + 4$

Then, $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

$f''(x) = 6x - 6 = 6(x-1)$



* Analyzing local maxima/minima & increasing/decreasing:

For local maxima/minima we need the critical points:

So, $f'(x) = 0 \Rightarrow 3(x-3)(x+1) = 0 \Rightarrow x = -1, 3$

Now we get intervals $(-\infty, -1)$, $(-1, 3)$, $(3, \infty)$ corresponding to the critical pts.

In $(-\infty, -1)$, $f'(x) > 0$ as -2 is in $(-\infty, -1)$ & $f'(-2) = 5 > 0$
Therefore, the function f is increasing in $(-\infty, -1)$

In $(-1, 3)$, $f'(x) < 0$ as 0 is in $(-1, 3)$ & $f'(0) < 0$
Therefore, the function f is decreasing in $(-1, 3)$.

In $(3, \infty)$, $f'(x) > 0$ as 5 is in $(3, \infty)$ & $f'(5) = 12 > 0$
Therefore, the function f is increasing in $(3, \infty)$

Hence, $f(x)$ is increasing in $(-\infty,-1) \cup (3,\infty)$ & decreasing in $(-1,3)$.

Also, on left to the critical point $x=-1$, f is increasing & on right to the critical point $x=-1$, f is decreasing

$\Rightarrow$ $x=-1$ is a maxima.

Again, on left to the critical point $x=3$, f is decreasing & on right to the critical point $x=3$, f is increasing.

$\Rightarrow$ $x=3$ is a minima.

⊛ Analyzing Concave Up/Down & Point of Infections:

For possible Point of Inflection:

$$f''(x)=0 \Rightarrow 6(x-1)=0 \Rightarrow x=1$$

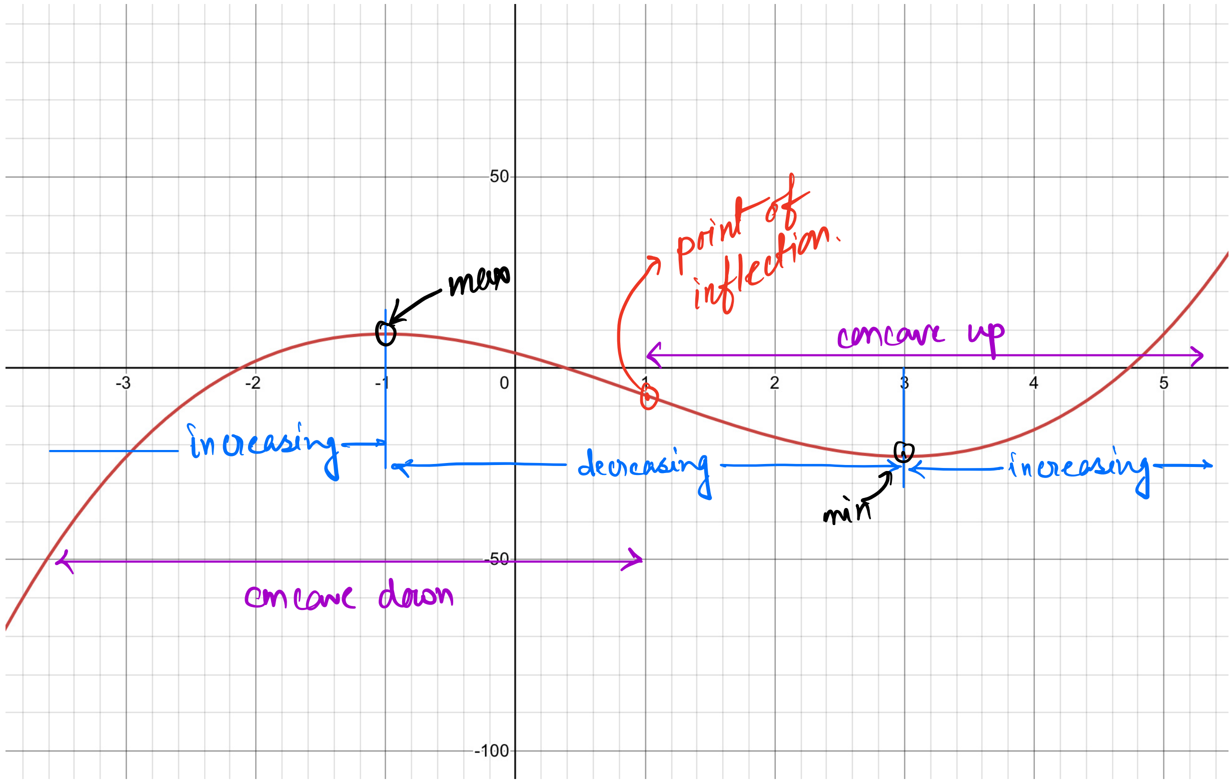
Now, we get intervals $(-\infty,1)$ & $(1,\infty)$ corresponding to the possible point of inflection.

In $(-\infty,1)$, $f''(x)<0$ as 0 is in $(-\infty,1)$ & $f''(0)<0$
Therefore, f is concave down.

In $(1,\infty)$, $f''(x)<0$ as 2 is in $(1,\infty)$ & $f''(2)>0$
Therefore, f is concave up.

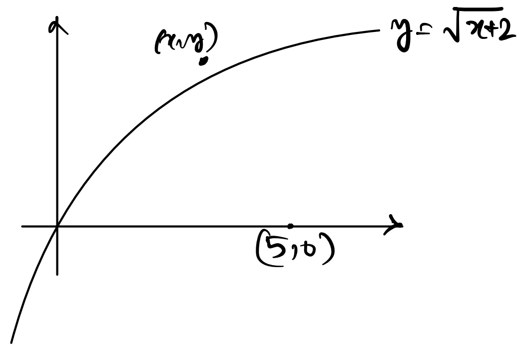
Since the sign is changing for f'' near $x=1$, it is a point of inflection.

Actual Graph:



Q. Find the closest point to $(5,0)$ on the curve $y=\sqrt{x+2}$, $x \geq -2$

Sol^n:- Choose a pt on $y=\sqrt{x+2}$. Say (X,Y).



Distance between (X,Y) & $(5,0)$ is

$$D=\sqrt{(X-5)^2+(Y-0)^2}$$

$$=\sqrt{(X-5)^2+Y^2}$$

Now, (X,Y) is a point on $y=\sqrt{x+2}$, so

$$Y=\sqrt{X+2}$$

Then $D = \sqrt{(x-5)^2 + y^2}$ can be written in one variable x as:

$$D(x) = \sqrt{(x-5)^2 + (\sqrt{x+2})^2}$$
$$= \sqrt{(x-5)^2 + (x+2)}$$
$$= \sqrt{x^2 - 10x + 25 + x + 2}$$
$$= \sqrt{x^2 - 9x + 27}.$$
$$= (x^2 - 9x + 27)^{1/2}$$

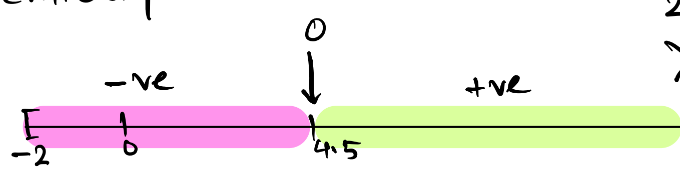
$$D'(x) = \frac{1}{2}(x^2 - 9x + 27)^{-1/2} \cdot (2x - 9)$$
$$= \frac{2x-9}{2(x^2 - 9x + 27)}$$

$D'(x) = 0$ or $D'(x)$ undefined gives

$D'(x) = 0$
$\Downarrow$
$2x - 9 = 0$
$\Rightarrow x = 4.5$ critical pt

$D'(x)$ undefined
$\Downarrow$
$x^2 - 9x + 27 = 0$
$\Rightarrow x = \dfrac{9 \pm \sqrt{81 - 4 \cdot 1 \cdot 27}}{2 \cdot 1} \approx \sqrt{-ve}$
$\times$

Note:- $D'(x)$



with increasing x, $D'(x)$ is increasing
$\Rightarrow D''(x) > 0$
$\Rightarrow D(x)$ has a minima.

So, $x = 4.5 \Rightarrow y = \sqrt{x+2} = \sqrt{6.5}$

Hence, $(4.5, \sqrt{6.5})$ is the closest pt on $y = \sqrt{x+2}$ from $(5, 0)$